RELATIONSHIP OF THE NEW TWO-PARAMETER FAMILY OF FLOWS WITH FREE BOUNDARIES TO KIRCHHOFF, ÉFROS, AND ZHUKOVSKII–ROSHKO FLOWS

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It is a well-known fact that the classical flow with free boundaries formulated by Kirchhoff in 1869 set the stage for the theoretical investigation of separated incompressible fluid flow around bodies. The complexity of the problem has been well perceived by researchers, who have therefore embraced the model approach, as nowhere else in fluid dynamics, in order to simplify the given problem to the maximum. It became clear in the thirties and forties that the model approach using flows with free boundaries permits the separation flow of an incompressible fluid around bodies to be described quite closely to the experimental results in fully developed cavitation regimes, where a cavity formed by flow separation is completely filled with vapor or gas, i.e., in application to two-phase fluid flows, but this approach does not afford a description of single-phase fluid flows that adequately represents the experimental results. Model flows with free boundaries were proposed during this period and could be used to describe flow around bodies for nonzero values of the cavitation number $Q = 2(p_\infty - p_0)/\rho v_\infty^2$ (p_0 is the pressure in the cavity), where $Q = 0$ corresponds to Kirchhoff flow, and it was regarded as the limiting flow state to which model flows should converge as $Q \to 0$. Indeed, the values of the drag coefficient c_x determined according to the Zhukovskii–Roshko, Ryabushinskii, and Éfros schemes for a flat plate oriented perpendicular to the direction of the freestream velocity differ somewhat for $Q \neq 0$, tending to the Kirchhoff value $c_x = 2\pi/(\pi + 4)$ in the limit $Q \to 0$.

However, the transition from $Q = 0$ in Kirchhoff flow to $Q \neq 0$ required the insertion of additional bodies (plates) in the flow if the flow took place on one sheet of a Riemannian surface (Zhukovskii–Roshko and Ryabushinskii schemes) or the application of an auxiliary second sheet of the Riemannian surface (Éfros scheme); this violated the customary hydrodynamic statement of the problem of an unbounded ideal fluid flow around an isolated body and left unresolved the question of the additional (under the mutual influence of the target body and the streamline-closing bodies) increment or decrement of its drag c_x. These models also failed to account for the existence of vortex flow in the wake of a real cavity (vortex shedding), because, e.g., the generation of a return stream takes place in Éfros flow without regard for the influence of the displacement thickness of the vortex flow in the wake, despite the fact that the displacement thicknesses in a real cavitation flow and the transverse width of the body are of the same order of magnitude.

These considerations, along with other questions that have clouded the issue so far in connection with the unsteadiness (time dependence) of a real cavitation flow and the mechanism of kinetic energy dissipation of the fluid, motivated one of the present authors to describe cavitation flows by an energy approach, which permits a less detailed description of the flow than is required by the force approach, but takes into account the governing characteristics of a real flow, one of which is the existence of vortex shedding from a cavity.

In the model of a second dissipative layer and wake [1], the vortex flow in the vicinity of the wake is modeled by a displacement half-body, which is inserted into the potential flow behind the main body. The dimensionless thickness $\bar\delta_\infty^* = \delta_\infty^*/b$ of the displacement half-body at an infinite distance behind the main body is related to c_x by the equation $\bar\delta_\infty^* = \bar\delta_\infty^{**} = c_x/2$ ($\bar\delta_\infty^{**} = \delta_\infty^{**}/b$ is the dimensionless momentum loss thickness, and b is a characteristic width of the body).

Normally in applications of the model of a second dissipative layer and wake to problems of viscous fluid flow around bodies, the displacement half-body is reproduced by means of a source located on the aft side of the body [1, 2]. However, if the source is placed behind a

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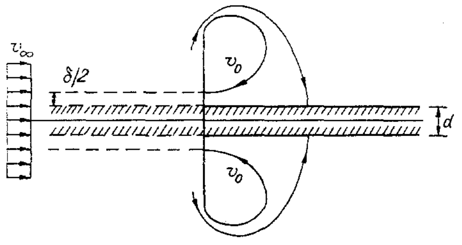

Fig. 1

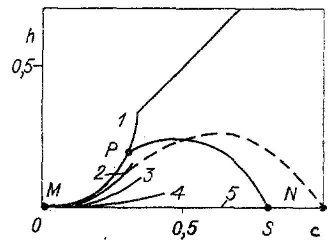

Fig. 2

cavity in the given cavitation flow situation, the actual process of return-stream generation can be distorted by the strong local perturbation created by the source. Clearly, the smallest local perturbation is introduced by a displacement half-body formed by two parallel plates separated by a distance $d = \delta_\infty^2$. Now, in addition to the well-known one-parameter family of Éfros flows (with the cavitation number Q as the parameter), it is also necessary to consider a new two-parameter family of flows (with independent parameters Q and d), which have two return streams located symmetrically about the x axis with a combined thickness δ [3].

Sadovskii [4] has investigated the solution of the flow problem according to a new scheme, where a plate is situated perpendicular to the direction of the freestream velocity (Fig. 1), determining the domain of admissible values of the independent mathematical parameters h and c (which are bound by a definite relation in Éfros flow and plotting analytical graphs of the total thickness of the return streams as a function of d for several values of Q.

If these quantitative data are introduced in the theory of quasisteady separated flow around bodies [3] formulated within the context of the model of a second dissipative layer and wake, and if the theoretical predictions are compared with the results of experiments on the value of c_x for a plate oriented perpendicular to the direction of the freestream velocity, the theoretical description is observed to match the experimental results adequately not only in the case of separated flows with fully developed cavitation ($0 < Q < 1$), but also for the separated flow of a single-phase fluid flow (according to the theory, the maximum possible value $c_x = 2.0$ practically coincides with the value of c_x in a single-phase fluid, which has long been known from experimental work).

In the present article we investigate the age-old question raised by Kelvin [5] in connection with classical Kirchhoff flow and discussed repeatedly in later studies (see, e.g., [6]), as to the physical applicability of mathematical flows with free boundaries for the description of separated flow around bodies.

The relationship between the new two-parameter family of flows with free boundaries and the well-known one-parameter Éfros and Zhukovskii–Roshko flows has been established previously [4] [either of these represents limiting (boundary) terms of the two-parameter family]. The upper curvilinear part 1 of the boundary $h = h(c)$ of the domain of the two-parameter family in the plane of the parameters h and c in Fig. 2 describes the family of Éfros flows, and the lower boundary $h \equiv 0$ describes Zhukovskii–Roshko flows. For a constant value of the cavitation number ($Q > 0$), as the distance between parallel plates is varied continuously from $d = 0$ to $d_{max}(Q)$ (points P and S in Fig. 2, which are associated with Éfros and Zhukovskii–Roshko flows, respectively), the operating point moves along the curve PS in the domain of the two-parameter family with a continuous variation of the total thickness of the return streams from $\delta_{max}(Q)$ to $\delta = 0$. Consequently, for $Q > 0$ the set of possible flows with finite thicknesses of the return flows and displacement half-bodies completely describes the investigated two-parameter family.

The following characteristics of the limiting transitions observed in [4] with respect to the parameters h and c to the point $h = C = 0$ ($Q \to 0$) are significant in the ensuing discussion:

1. As the origin (Fig. 2) is approached along the upper boundary 1 (i.e., within the framework of the family of Éfros flows, where $h = c^2 + c^3 + 3c^4 + \ldots$ as $c \to 0$), the results obtained in the limit for d, δ, and c_x are well known, viz.: Kirchhoff drag $c_x^* = 2\pi/(\pi + 4)$,

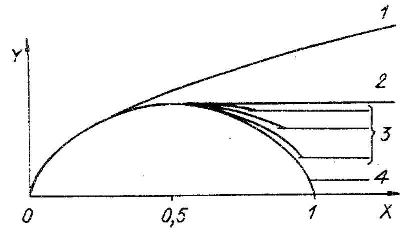

Fig. 3

a finite value of the stream thickness $\overline{\delta}_* = \pi/[2(\pi + 4)]$, and zero thickness of the wake d = 0.

2. The result of passage to the limit along the lower boundary 5 (h ≡ 0, c → 0) is also well known: $\delta = 0$, $\overline{d} \approx \pi/[2c(\pi + 4)] \to \infty$, $c_x = c_x^* = 2\pi/(\pi + 4)$.

3. If $h = c^2 + \beta c^3 (\beta < 1)$ and c → 0 (curve 2, which has the same curvature as the Éfros boundary 1), then $cc_x = 2\pi/(\pi + 4)$ in the limit, but the two return streams are separated by a wake of finite thickness $\overline{d} = \pi(1 - \beta)/[2(\pi + 4)]$, which is determined by the coefficient β. The total thickness of the streams, on the other hand, is equal to the Éfros limiting value $\overline{\delta} = \overline{\delta}_* = \pi/[2(\pi + 4)]$.

4. If $h = \alpha c^2$ ($0 < \alpha < 1$; curve 3 in Fig. 2, which has a smaller curvature than 1), the same Kirchhoff drag $c_x = 2\pi/(\pi + 4)$ is obtained in the limit c → 0, but the total thickness of the return streams is $\overline{\delta} = \pi\alpha^2/[2(\pi + 4)]$ and can assume values in the interval $0 < \overline{\delta} < \overline{\delta}_*$ depending on α. The distance between the streams is $\overline{d} = \pi(1 - \alpha)/[2c(\pi + 4)] \to \infty$, but the ratio of d to the principal cavity diameter is finite and smaller than unity in the limit.

5. The passage to the limit c → 0 under the condition $h \approx c^\gamma (\gamma > 2)$, i.e., along curve 4, which is next to the Zhukovskii–Roshko boundary 5 in curvature, also gives a result similar to case 2: $c_x = c_x^* = 2\pi/(\pi + 4)$.

These data not only indicate the abundant content of the two-parameter family at the singular point h = c = 0, but also afford the possibility of comparing these limiting flows both with each other and with the classical Kirchhoff flow (Q = 0); such a comparison has been excluded from any consideration to date.

It was remarked above that in all the limiting (Q → 0) cases of flow around a plate oriented perpendicular to the freestream direction its drag is the same and equal to the drag on a plate in Kirchhoff flow.

A visual picture of the difference in the global geometrical pattern of the flows can be obtained by analyzing them in the compressed variables $X = x/L$, $Y = y/\sqrt{L}$, where L is the longitudinal dimension of a cavity in Éfros flow in the limit Q → 0, referred (like x and y) to the length of the plate. The boundaries of the limiting flows are shown qualitatively in these variables in Fig. 3, viz.: 1) the Kirchhoff family $Y \sim \sqrt{X}$; 2) the Zhukovskii–Roshko family; 3) the two-parameter family in the limiting transition 4; 4) the Éfros family. Naturally, the plate shrinks to a point in these coordinates, and the return streams are reduced to lines, segments of which are shown in Fig. 3. We see that the mass of the rest fluid in Kirchhoff flow is infinitely times as great as the analogous mass in all the limiting flows; of these, Éfros flow has the minimum mass. There is one other distinctive feature to consider. The drag on the plate in Kirchhoff flow is associated with momentum losses in the external flow. In the Éfros flow limit, exactly the same drag is attributable to the formation of a return stream. In intermediate limiting flows of the two-parameter family (line 3 in Fig. 3) the same drag is now created by the action of both factors.

Consequently, not only the difference in the flow geometry observed in Fig. 3, but also the difference in the physical mechanism responsible for the creation of drag makes the Éfros flow irreducible to Kirchhoff flow in the limit Q → 0; an infinite number of flows occurs between them with a smaller total thickness of the return stream than in Éfros flow, but with the same drag. For this reason, it is impossible to concur with the viewpoint taken in the literature on cavitation flows, that as Q → 0 Éfros flow goes over to Kirchhoff flow possessing a nonvanishing return stream with its base infinitely far away. Moreover, the investigation of the properties of the two-parameter family of flows with free boundaries and the results of the theory of quasisteady separated flow around bodies permits a new approach to the age-old question as to the applicability of mathematical flows with free boundaries for the description of separated flows and jet flows; Kelvin raised this question back in 1894, at which time [5] (see also [6, 7]) he expressed skepticism about the possibility of describing separated flow around bodies theoretically by means of mathematical flows with free boundaries (with specific reference to Kirchhoff flow), but regarded the free-boundary flow introduced by Helmholtz in 1968 as physically applicable for the theoretical description of processes associated with the flow and interaction of fluid jets.

Kelvin saw the main flaw of Kirchhoff flow in the fact that the "dead water" following a plate of unbounded mass must have infinite kinetic energy. Despite the subsequent repeated contention of other authors (see, e.g., [6]) with Kelvin's reasoning, the results of model studies, including those cited in the present study, vindicate Kelvin and attest to the power of his physical foresight. The inapplicability of Kirchhoff global flow for the description of separated flow around bodies because of the excessive volume of the "dead water" and the applicability of flows with free boundaries for describing the flow and interaction of jets have been corroborated. First, it is evident from Fig. 3 that the volume of the detached zone in Éfros flow is infinitely times as small as the detached zone in Kirchhoff flow; second, the drag acting on the body in Éfros flow is created by the flow of the return stream and the associated loss of momentum, and not by the loss of momentum in the external flow as in the case of Kirchhoff flow.

The latter fact has proved to be very important in the energy approach to the theory of separated flow around bodies insofar as the drag-induced dissipation of kinetic energy of the return stream could be incorporated into the model. The consistency of the drag coefficient governed by the dissipation of kinetic energy of the return stream and the thickness of the displacement half-body used to model vortex shedding ensures that the description of the behavior of c_x or a plate in a two-phase fluid will adequately represent the experimental data and that will have the correct value in a single-phase fluid, and it allows the flow field to be described reliably with increasing distance from the symmetry plane.

The model requirements on the consistency of c_x for a plate, dissipation, and the displacement thickness of the vortex wake are satisfied on the dashed curve MN on the plane of the mathematical parameter (h, c) in Fig. 2. As the operating point moves along this curve from point M to point N, all the model flows corresponding to a variation of the cavitation number from $Q = 0$ to $Q = \infty$ are traversed accordingly in the physical plane. It is interesting to note that the line of the flows admitted by the model lie entirely within the domain of the new two-parameter family of flows and nowhere coincides with the Éfros and Zhukovskii–Roshko one-parameter flow families as the latter are approached.

Thus, aside from physical reality, the Éfros and Zhukovskii–Roshko mathematical flows, and not just the mathematical flow of Kirchhoff, remain in the model description of separated flow around bodies. However, Kirchhoff flow with the wake expansion law $Y = \sqrt{X}$ preserves its fundamental status in fluid dynamics, because it is known [8] to determine the limiting, physically admissible rate of expansion of a half-body in an unbounded plane-parallel inviscid incompressible fluid flow. Consequently, in the modeling of nonseparated viscous vortex flow around bodies by potential flows augmented by displacement half-bodies, as is done in applications of the model of a second dissipative layer and wake to highly viscous flows in the limit $Re \to 0$ [9], the limiting physically admissible rate of expansion of the displacement half-body is determined by the law $Y \sim \sqrt{X}$, i.e., by the law of expansion of the "dead water" zone in Kirchhoff flow. The "dead water" volume, which is too large for separated flows with finite values of c_x for a body in the flow and has prevented the application of Kirchhoff flow for the description of separated flows since Kelvin's time, does not preclude the physical realization of the model flow after an infinite time in the limit $Re \to 0$, because the drag coefficient c_x of the body in the planar problem of viscous fluid flow tends to infinity as $Re \to 0$.

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